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COEFFICIENTS OF NONVANISHING FUNCTIONS IN H^∞

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ABSTRACT

Let B denote the class of functions analytic in the unit disc of C which satisfy $0<|f(z)|<1$. It is proved that there exists a number $c<1$ such that if $f\in B$ and if $f(z)=\Sigma_{n=0}^{\infty} a_n z^n$, then $|a_n|<c$ for $n\geqq 1$.

1. Introduction

Let B denote the class of functions analytic in the unit disc of C which satisfy

$$0<|f(z)|<1 \quad \text{for } |z|<1.$$

We consider the coefficient problem for B, as posed by Krzyz [4]; that is, we seek

$$c_n=\sup_{f\in B}|f^{(n)}(0)/n!|;\quad n=1,2,\cdots.$$

It has been conjectured that $c_n=2/e$ for all n, with extremal function

$$f_n(z)=\exp\left\{\frac{z^n+1}{z^n-1}\right\}.$$

This conjecture has been verified up to $n=4$ (see [2] and [3]). For general n, much less is known. However, some information is available. For example, since B is a normal family of analytic functions, we are assured that the maximum c_n is attained for each n. More significantly, the form of the extremal functions is known; indeed, Atzmon [1] and Hummel, Scheinberg and Zalcman [3] have shown that

$$c_n=|f^n(0)/n!|$$

for some f of the form

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$$f(z) = \exp\left\{-\sum_{k=1}^{n} a_k \frac{e^{i\theta_k}+z}{e^{i\theta_k}-z}\right\}, \tag{1}$$

where the a_k are nonnegative numbers, and $\{e^{i\theta_k}\}$ are various points on the unit circle. In the language of Hardy space theory, the f which attains c_n is a singular inner function whose associated measure has at most n support points. Henceforth we shall restrict all of our attention to such functions.

From the formula

$$f^{(n)}(0)/n! = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta \tag{2}$$

it follows immediately that $c_n \leqq 1$ for all n. Somewhat surprisingly, this is the best *uniform* estimate of the c_n which has so far been obtained. Our purpose in the present article is to show that indeed there exists a number $c < 1$ such that

$$c_n < c \quad \text{for all } n.$$

Our method is to estimate, more or less directly, the integral in (2) when f is of the form (1). It appears that this technique cannot yield a sharp value for c, and so we have not expended undue effort refining it.

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2. The results

Our theorem will proceed quite easily from two lemmas. We state them at the outset.

LEMMA 1. *Let f be as in* (1), *and let* $\exp\{ih(\theta)\} = f(e^{i\theta})$ *denote the "boundary function" of f. Thus, in particular, h is real-valued. Define the sets*

$$K_1 = \left\{\theta : 0 \leqq \theta < 2\pi \text{ and } h'(\theta) > \frac{3}{2}n\right\}$$

and

$$K_2 = \left\{\theta : 0 \leqq \theta < 2\pi \text{ and } h'(\theta) < \frac{1}{2}n\right\}.$$

Then at least one of these sets is of Lebesgue measure greater than 2/3.

The significance of Lemma 1 becomes clear if we turn our attention to formula

(2). The integral which we wish to estimate there is simply that of $\exp\{i[h(\theta)-n\theta]\}$, and in view of Lemma 1, the exponent $h(\theta)-n\theta$ has a large derivative on a sizable portion of the circle. Such integrals are estimated in the following result, which generalizes a well-known lemma of van der Corput ([5], p. 197).

LEMMA 2. *Let f be a smooth real-valued function on the interval $[a,b]$. Assume that f'' is of constant sign on $[a,b]$ and that $|f'|>M>0$ throughout the interval. Then*

$$\left|\int_a^b \exp[if(x)]dx\right| \leqq \begin{cases} 2/M \ \textit{always}, \\ 2/M \sin \dfrac{(b-a)M}{2} \quad \textit{if} \quad M(b-a)<\pi. \end{cases}$$

For the sake of clarity of the exposition, it seems wise to prove Lemma 2 before Lemma 1. We turn to that proof immediately.

First, let us note that if

$$\int_a^b \exp[if(x)]dx = Re^{i\alpha},$$

then

$$\left|\int_a^b \exp[if(x)]\,dx\right| = R = \int_a^b \exp(i[f(x)-\alpha]) = \int_a^b \cos[f(x)-\alpha]\,dx,$$

and so, without changing the hypotheses, we might as well bound real integrals of the form $\int_a^b \cos f(x)dx$.

Second, we may assume without loss of generality that f' and f'' are positive on $[a,b]$. Indeed, if $f'<0$, we simply consider $-f$. If $f''<0$, we replace $f(x)$ by $-f(b-x)$.

Having made the above normalizations, let $x=g(y)$ be the inverse function of f. Then

$$\int_a^b \cos f(x)dx = \int_c^d \cos y\, g'(y)dy,$$

where $c=f(a)$ and $d=f(b)$. We claim that our last integral is majorized by

$$\left|\int_I \cos y\, g'(y)dy\right| \tag{3}$$

where I is some subinterval of $[c,d]$ on which $\cos y$ does not change its sign. If $\cos y$ is of constant sign on $[c,d]$, the claim is trivial. If not, we write

$$\int_c^d \cos y\, g'(y)dy = \int_c^{(k+(1/2))\pi} \cos y\, g'(y)dy$$
$$+\sum_{j=k}^{l-1}\int_{(j+(1/2))\pi}^{(j+(3/2))\pi} \cos y\, g'(y)dy + \int_{(l+(1/2))\pi}^{d} \cos y\, g'(y)dy.$$

Since g' is positive and decreasing, the above series is alternating in sign, and decreasing after the first term. Thus it is majorized by one of its terms, and our claim is proved.

From the hypothesis that $f' > M$ on $[a, b]$, we conclude that

$$0 < g'(y) < \frac{1}{M}; \quad y \in [c, d],$$

and it follows immediately that the expression (3) is less than or equal to 2/M, proving one assertion of the lemma.

Now, let us assume that $M(b-a) < \pi$, and we shall seek a refined estimate for (3). In light of the inequality

$$\int_I g'(y)dy \leqq \int_c^d g'(y)dy = b - a,$$

we may view the integral

$$\int_I \cos y\, g'(y)dy$$

as a weighted average with total mass $\leqq (b-a)$ of the values of $\cos y$ on I. Clearly such an average is greatest when the maximal density ($1/M$ in our case) is placed on the largest part of the function $\cos y$; that is, when $g' = 1/M$ on an interval of length $(b-a)M$. We conclude that

$$\left| \int_a^b \cos f(x)dx \right| \leqq \frac{1}{M}\int_{-((b-a)/2)M}^{((b-a)/2)M} \cos y\, dy = \frac{2}{M}\sin\left(\frac{b-a}{2}M\right). \quad \text{Q.E.D.}$$

We turn now to the proof of Lemma 1. Our interest is directed at the function h defined by

$$\exp[ih(\theta)] = f(e^{i\theta}),$$

with f as in (1). A simple calculation shows that

$$h(\theta) = -\sum_{k=1}^{n} a_k \operatorname{ctg}\left[\frac{1}{2}(\theta - \theta_k)\right].$$

It is perhaps worthwhile to remark here that although we write $h(\theta)$ and not $h(e^{i\theta})$, we relate to h as a function on the circle; that is, we identify the endpoints of the θ-interval $[0,2\pi]$. Now

$$h'(\theta)=\sum_{k=1}^{n}\frac{1}{2}a_k\csc^2\left[\frac{1}{2}(\theta-\theta_k)\right]. \tag{4}$$

an expression in which every term is positive on $[0,2\pi]$. Similarly, one finds that h''' is always positive. From these observations, there emerges a rough sketch of the behavior of h' and h''. First of all, between any two successive points θ_k, h'' rises continually from $-\infty$ to $+\infty$. Meanwhile, on such an interval, h' falls from $+\infty$ to some positive minimum, after which it rises back to $+\infty$.

We proceed to investigate the set K_1 mentioned in the statement of Lemma 1. We shall (harmlessly) include in K_1 the points $\{\theta_k\}$ where $h'=\infty$. With this convention, the above description of h' shows that K_1 is an open subset of the circle, consisting of at most n disjoint open intervals $I_1,\cdots,I_s$. Moreover, each I_j contains at least one of the points $\{\theta_k\}$.

We wish to estimate the length of a typical I_j in terms of those θ_k's which it contains. If this length is greater than 2/3, the conclusion of Lemma 1 is already fulfilled. If not, we can write I_j as the interval (α_j,β_j), with $\beta_j-\alpha_j<2/3$. But

$$\frac{3n}{2}=h'(\beta_j)\geqq\sum_{\theta_k\in I_j}\frac{1}{2}a_k\csc^2\left[\frac{1}{2}(\beta_j-\theta_k)\right]\geqq\csc^2\left[\frac{1}{2}(\beta_j-\alpha_j)\right]\sum_{\theta_k\in I_j}\frac{1}{2}a_k.$$

Thus

$$\frac{1}{\sqrt{3n}}\left(\sum_{\theta_k\in I_j}a_k\right)^{1/2}\leqq\sin\left[\frac{1}{2}(\beta_j-\alpha_j)\right]\leqq\frac{1}{2}(\beta_j-\alpha_j).$$

Summing over j, we obtain an estimate of the measure of K_1, namely

$$|K_1|\geqq\frac{2}{\sqrt{3n}}\sum_j\left(\sum_{\theta_k\in I_j}a_k\right)^{1/2}. \tag{5}$$

We shall now show that if $|K_1|$ is small, then h' is "often" smaller than $n/2$; that is, $|K_2|$ is large. In particular, we shall estimate $\int_{T-K_1}h'(\theta)d\theta$, where T is the whole interval $[0,2\pi]$. According to formula (4), this integral equals

$$\begin{aligned}&\int_{T-K_1}\sum_k\frac{1}{2}a_k\csc^2\left[\frac{1}{2}(\theta-\theta_k)\right]d\theta\\&\leqq\sum_j\sum_{\theta_k\in I_j}\int_{T-I_j}\frac{1}{2}a_k\csc^2\left[\frac{1}{2}(\theta-\theta_k)\right]d\theta\\&=\sum_j\sum_{\theta_k\in I_j}\left|a_k\operatorname{ctg}\left[\frac{1}{2}(\beta_j-\theta_k)\right]-a_k\operatorname{ctg}\left[\frac{1}{2}(\alpha_j-\theta_k)\right]\right|\end{aligned}$$

where α_j and β_j are the endpoints of I_j, $j = 1, 2, \cdots, s$. Thus, by Schwarz's inequality,

$$\int_{T-K_1} h'(\theta)d\theta$$

$$\leqq \sum_j \Big(\sum_{\theta_k \in I_j} a_k\Big)^{1/2} \Big[\Big(\sum_{\theta_k \in I_j} a_k \operatorname{ctg}^2\Big[\frac{1}{2}(\beta_j - \theta_k)\Big]\Big)^{1/2} + \Big(\sum_{\theta_k \in I_j} a_k \operatorname{ctg}^2\Big[\frac{1}{2}(\alpha_j - \theta_k)\Big]\Big)^{1/2}\Big].$$

Since $\operatorname{ctg}^2(\alpha) < \csc^2(\alpha)$ for all α, it follows from (4) that our last expression is dominated by

$$\sum_j \Big(\sum_{\theta_k \in I_j} a_k\Big)^{1/2} [(2h'(\beta_j))^{1/2} + (2h'(\alpha_j))^{1/2}] = 2\sqrt{3n} \sum_j \Big(\sum_{\theta_k \in I_j} a_k\Big)^{1/2}. \tag{6}$$

We now distinguish between two cases. In the first, $|K_1| > 2/3$, and the claim of the lemma holds. In the second case, $|K_1| \leqq 2/3$. According to (5), it follows that

$$\sum_j \Big(\sum_{\theta_k \in I_j} a_k\Big)^{1/2} \leqq \frac{\sqrt{3n}}{3}.$$

Hence the estimate (6) implies that

$$\int_{T-K_1} h'(\theta)d\theta \leqq 2n.$$

Since $T - K_1$ is of measure at least $2\pi - 2/3$, one deduces immediately that $|K_2| > 2/3$. This completes the proof of Lemma 1.

It is now quite easy to obtain our desired estimate of $c < 1$ for the integral

$$\frac{1}{2\pi} \int \exp(i[h(\theta) - n\theta])d\theta.$$

Writing $g(\theta) = h(\theta) - n\theta$, we have seen that $|g'(\theta)| > n/2$ on some set K of measure $|K| = 2/3$. Since $g'' = h''$, whose behavior we have analyzed, and since we may take K to be a subset of either K_1 or K_2 as constructed in Lemma 1, we may assume that K is a union of at most $2n$ intervals $J_1, \cdots, J_r$ on which g'' is finite and of constant sign. Let us index these intervals J_k so that

$$|J_k| < 2\pi/n; \quad k = 1, 2, \cdots, q,$$

and

$$|J_k| \geqq 2\pi/n; \quad k = q+1, q+2, \cdots, r.$$

According to Lemma 1,

$$\left|\int_K \exp[ig(\theta)]d\theta\right| \leqq \sum_{k=1}^{r}\left|\int_{J_k} \exp[ig(\theta)]d\theta\right|$$

$$\leqq \sum_{k=1}^{q} \frac{4}{n}\sin\left(\frac{n}{4}|J_k|\right) + \frac{2}{\pi}\sum_{k=q}^{r}|J_k|,$$

and this is less than or equal to

$$\frac{4q}{n}\sin\left(\frac{n}{4q}\sum_{k=1}^{q}|J_k|\right) + \frac{2}{\pi}\sum_{k=q}^{r}|J_k|, \tag{7}$$

by the concavity of the function $\sin x$ on $[0, \frac{1}{2}\pi]$.

It is an exercise in differential calculus to show that subject to our conditions

$$\sum_{k=1}^{r}|J_k| = 2/3$$

and

$$|J_k| < \frac{2\pi}{n}, \quad k = 1, 2, \cdots, q,$$

the expression (7) will be largest when $q = r$, and when q is as large as possible, namely $q = 2n$. One concludes that

$$\left|\int_K \exp[ig(\theta)]d\theta\right| \leqq 8\sin\left(\frac{1}{12}\right) < \frac{2}{3} = |K|.$$

Putting together all of the pieces, we have now proved the following result:

THEOREM. *If $f \in B$, and if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then*

$$|a_n| \leqq 1 - \frac{1}{3\pi} + \frac{4}{\pi}\sin\left(\frac{1}{12}\right).$$

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